Letters

An Alternate Derivation of Lewin's Formula

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Abstract—**The reflection coefficient at the junction between a uniform waveguide and one which tapers linearly in the plane of the electric field can be evaluated by a suitable integration of the differential reflection coefficient. When the limit of this infinite integral is determined as the "flare" angle approaches zero, the relative susceptance at the discontinuity is found to be the well-known expression** $B/Y_0 = \tan\theta/kb$.

Introduction

Interest in the reflection coefficient which results when a rectangular waveguide flares abruptly in the E-plane arises in two different applications. The first is the design of electromagnetic horns where this reflection is undesirable and where it was soon determined that it could be cancelled out by means of an inductive iris placed at the discontinuity. Lewin [1] has derived a very useful formula for the lumped capacitance appearing at this type of discontinuity. By requiring that the electric and magnetic fields at the center of the junction be continuous for the fundamental modes of the waveguide on one side of the junction and the radial outgoing mode on the other side of the junction, he has obtained an expression for the relative shunt capacitance introduced by the discontinuity which is given by the expression

$$B/Y_0 = \tan\theta/kb$$

where θ is the flare angle shown in Fig. 1, b is the height of the waveguide, and $k = 2\pi/\lambda_g$.

Rice [2] and Leonard and Yen [3] have obtained the same result in different ways while considering the general solution of this electromagnetic boundary value problem.

The second application in which the reflection coefficient is of interest is the design of continuous transformers between waveguides of different heights. Those interested in this problem have usually approached it as a transmission-line problem. They have used the notion of a differential reflection coefficient, proposed by Walker and Wax [4], and determined the reflection coefficient at the input to the transformer by integrating the differential coefficient over the length of the transformer, taking into account the proper phase factor. Matsumoru [5] has determined experimentally the VSWR's for a two-to-one linear transformer over a range of taper lengths and found them to agree well with values obtained by this method. On the other hand, this author has found that these experimental values also agree well with those obtained using Lewin's formula, assuming that the only reflections which occur in the transformer are those which occur at the discontinuities.

It is natural then to inquire whether or not Lewin's formula can be obtained from the transmission-line point of view by integrating a suitable differential coefficient. It is the object of this letter to show that this can be done.

Consider Fig. 1 in which the height of the waveguide is b_0 for $x < 0$ and $b_0 + 2\tan\theta x$ for $x > 0$. If the impedance of the flared section is normalized to the uniform waveguide,

$$Z(x) = 1 + \frac{2\tan\theta}{b_0}x, \qquad x > 0. \tag{1}$$

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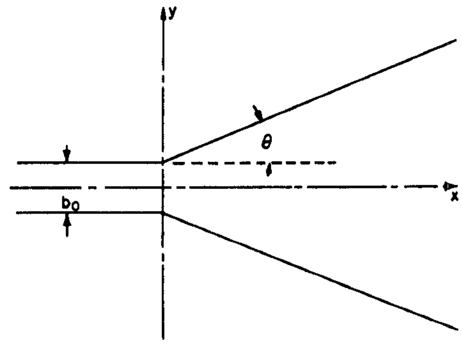

Fig. 1.

Now the differential coefficient $d\Gamma$ is given by

$$d\Gamma = \frac{Z(x+dx) - Z(x)}{Z(x+dx) + Z(x)} = \frac{\tan\theta\, dx}{b_0\left(1 + 2\dfrac{\tan\theta}{b_0}x\right)}. \tag{2}$$

Then the reflection coefficient at the origin is

$$\Gamma = \frac{\tan\theta}{b_0}\int_0^\infty \frac{e^{-j2kx}\, dx}{1 + \dfrac{2\tan\theta}{b_0}x} \tag{3}$$

where $k = 2\pi/\lambda_g$.

If the change of variable $y = 2kx$ is made,

$$\Gamma = \frac{\tan\theta}{2kb_0}\int_0^\infty \frac{e^{-jy}\, dy}{1 + \dfrac{\tan\theta}{kb_0}y}. \tag{4}$$

Then setting $L = \tan\theta/kb_0$,

$$\Gamma = \frac{L}{2}\int_0^\infty \frac{e^{-jy}\, dy}{1 + Ly}. \tag{5}$$

The Problem

The problem is the determination of the limiting value of this infinite integral as $L \to 0$. Clearly the limit is indeterminate if we simply put $L = 0$ in the denominator of (5). Now integrating by parts[1]

$$\int_0^\infty \frac{e^{-jy}\, dy}{1 + Ly} = \left.\frac{e^{-jy}}{-j(1+Ly)}\right|_0^\infty - \int_0^\infty \frac{-Le^{-jy}\, dy}{-j(1+Ly)^2}$$

$$= -j + jL\int_0^\infty \frac{e^{-jy}\, dy}{(1+Ly)^2}. \tag{6}$$

The integral is obviously bounded even for $L = 0$ thus

$$\lim_{L\to 0}\int_0^\infty \frac{e^{-jy}\, dy}{1 + Ly} = -j. \tag{7}$$

Finally, the reflection coefficient Γ to the first order in L is found to be

$$\Gamma = -j\frac{L}{2}. \tag{8}$$

[1] The author is indebted to the reviewers, both of whom observed that the original proof of (8), which consisted of eight pages and 33 equations, could be reduced to one or two lines They also pointed out how higher order terms could be obtained

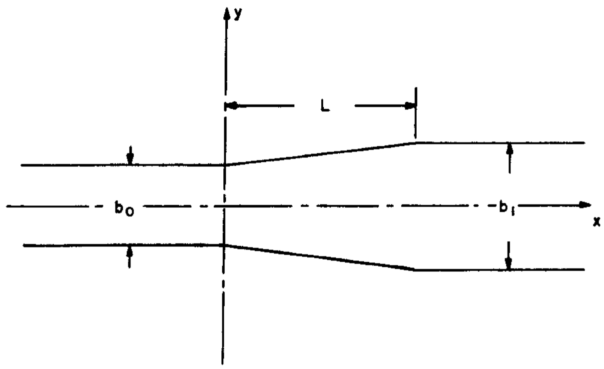

Fig. 2.

This is precisely what one would find from Lewin's formula for small L.

If this procedure is continued, one obtains

$$\Gamma \simeq -j\frac{L}{2}\sum_{n=0}^{\infty} j^n n!\, L^n. \qquad (9)$$

This representation, although divergent for all values of L, represents Γ asymptotically for small values of L.

REFERENCES

[1] L. Lewin, "Reflection cancellation in waveguides," *Wireless Engineer*, pp. 258–264, Aug. 1949.
[2] S. O. Rice, "A set of second order differential equations associated with reflections in rectangular waveguides—Application to guide connected to horn," *Bell Syst. Tech. J.*, vol. 28, pp. 136–156, 1949.
[3] D. J. Leonard and J. L. Yen, "Junction of smooth flared waveguides," *J. Appl. Phys.*, vol. 28, no. 12, pp. 1441–1448, Dec. 1957.
[4] L R. Walker and N Wax, "Nonuniform transmission lines and reflection coefficients," *J. Appl. Phys.*, vol. 17, pp. 1043–1045, 1946.
[5] K. Matsumaru, "Reflection coefficient of *E*-plane tapered waveguides," *IRE Trans. Microwave Theory and Tech.*, pp. 143–149, Apr. 1958.

Correction to "Analytical IC Metal-Line Capacitance Formulas

W. H. CHANG

In the above paper,[1] equation (13) should have read

$$r_b \approx \eta + \frac{p+1}{2}\ln \Delta. \qquad (13)$$

Equations (11)–(15) are valid for $w/h \geq 5$. If $5 > w/h \geq 1$, r_b should be iterated once by substituting that obtained from (13) into the following iteration formula:

$$r_b = r_b - \sqrt{(r_b-1)(r_b-p)} + (p+1)\tanh^{-1}\sqrt{\frac{r_b-p}{r_b-1}} - 2p^{1/2}\tanh^{-1}\sqrt{\frac{r_b-p}{p(r_b-1)}} + \frac{\pi W}{2h}p^{1/2}$$

which was inadvertently omitted. The iterated r_b is then substituted in (11) to obtain the capacitance.

The last line on p. 610 should have read "... for $w/h \geq 0.5$ and $w/d \geq 0.5$."

The previous errors should be correspondingly corrected in the Summary Section.

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[1] W. H. Chang, *IEEE Trans. Microwave Theory Tech.* (Lett.), vol. MTT-24, pp. 608–611, Sept. 1976.

Correction to "Electromagnetic Fields Induced Inside Arbitrary Cylinders of Biological Tissue"

T. K. WU AND L. L. TSAI

In the above paper,[1] Fig. 8 is not correct. The corrected Fig. 8 is presented here.

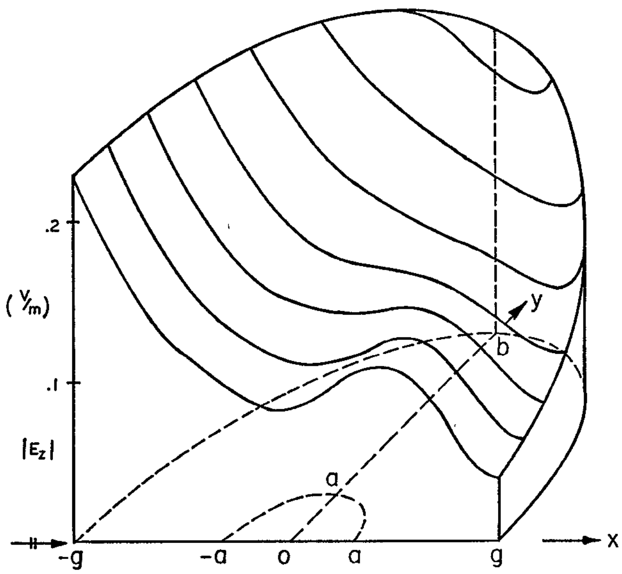

Fig. 8. *E*-field plot for a composite dielectric cylinder of elliptical muscle ($\varepsilon_2 = 55$, $\sigma_2 = 1.1$ ℧/m, $g = 6.5$ cm, $b = 9$ cm, $f = 300$ MHz) encasing circular bone ($\varepsilon_3 = 6$, $\sigma_3 = 0.04$ ℧/m, $a = 2$ cm, TM incident wave).

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[1] T. K Wu and L. L. Tsai, *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 61–65, Jan. 1977.